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Reply by Author to W. H. Wittrick

J. S. PRZEMIENIECKI*

Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio

THE discussion of current work by Wittrick and his associates at Birmingham University is of a great interest. Not many exact solutions to complicated buckling problems are available and this work will be helpful in comparing the finite element solutions with those obtained by solving the nonlinear equations for elastic thin-walled plate structures subjected to compressive loading.

Wittrick's comment that the finite element stiffness in Ref. 1 is a special case of his exact stiffness calls for a further clarification in order to put it in proper perspective. Coefficients in the approximate stiffness matrices could, in fact, be derived by expanding the exact coefficients represented by a quotient of two complicated transcendental functions of the compressive stress σ and the half-wavelength λ and retaining only the first order terms, a monumental task in exercising one's algebraic dexterity. Only through such a manipulation can any connection between Wittrick's stiffnesses and the finite element elastic and geometric stiffnesses be established. It should also be pointed out that the two methods are based on entirely different approaches. Wittrick uses an analytical solution of the nonlinear theory of elasticity while Ref. 1 uses the standard finite element approach based on the concept of geometrical stiffness. Thus, the author's method is simply an extension of the conventional finite element technique to a special class of problems involving local instability. There is still another fundamental difference between the two methods: Wittrick's method uses the stability determinant to obtain the buckling stress which contrasts with the author's use of the standard eigenvalue equations.

The author's comment that the computational procedure for obtaining the buckling stress from the exact solution is very time consuming was simply made on the basis of the comparison of typical elements in the stiffness matrices of the two methods. For example, the stiffness coefficient for the out-of-plane plate edge rotation in the exact method, using notation of Ref. 2, is given by

$$s_{MM} = \frac{D(\xi)^{1/2}}{b} \frac{\alpha \cosh \alpha \sinh \gamma - \gamma \cosh \gamma \sinh \alpha}{\sinh a \sinh \gamma + (\alpha\gamma/\omega^2)(1 - \cosh \alpha \cosh \gamma)} \quad (1)$$

for $\xi < 1$ while for $\xi > 1$ and $\xi = 1$ other similar transcendental expressions are used. In these expressions ξ, α, and γ are functions of the stress σ and the half-wavelength λ and ω depends on only λ. Equation (1) may be compared with the corresponding finite element stiffness coefficient expressed in algebraic form as

$$k_{22} = 2D\left[\frac{\pi^4}{420}\left(\frac{b}{\lambda}\right)^3 + \frac{\pi^2}{15}\left(\frac{b}{\lambda}\right) + \left(\frac{\lambda}{b}\right) + \frac{\pi^2 \sigma t b^2}{420D}\left(\frac{b}{\lambda}\right)\right] \quad (2)$$

where the first three terms belong to the elastic stiffness matrix and the fourth term is the geometrical stiffness. Dimensionally, Eq. (2) is different from that derived by Wittrick [Eq. (1)] because his expression is for a stress-couple (in lb-in./in.) while the finite element stiffness is for a node moment (in lb-in.). Noting that an iterative solution is required to determine the half-wavelength λ for the lowest stress σ in both the exact and finite element formulations of local instability analysis, the computational simplicity of the finite element coefficients, as typically represented by Eq. (2), is quite obvious.

Reference 1 gives typical computer times for the finite element computations, but unfortunately the corresponding times for the exact method are not available to make a meaningful comparison. It should be pointed out, however, that for many problems, the finite element solutions where each component flat is treated as a single element give sufficient accuracy for engineering purposes. Furthermore, since the author's method is based on the concept of geometrical stiffness, the method may be used directly in conjunction with any finite element computer programs for the over-all instability—a definite advantage for the design engineer who needs to investigate not only the local instability but also the over-all instability of the same configuration.

The restrictive assumption that the edge lines must remain straight during buckling (the classical assumption for local instability) used in Ref. 1 can be removed and the finite element method, developed by the author, can be extended to include the in-plane stiffnesses. In fact, work is presently underway to develop the necessary matrices. This extension would permit finite element studies of coupling between long wave and short wave (local) modes of instability which can presently be accomplished with Wittrick's exact method.

Finally, regarding the comment that some of the numerical results in Fig. 5 of Ref. 1 should be above the exact curve, it should be reaffirmed that the numerical results are correct. When the biaxial stress field is compressive (both σ_x and σ_y are positive) the convergence of finite element solutions is from above; however, when the σ_y stress is tensile (negative σ_y) and it is sufficiently high in relation to σ_x, the convergence is from below. The stabilizing influence of the tensile stress in the transverse direction on the strip apparently has a dominant effect on the convergence.

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Index categories: Aircraft Structural Design (Including Loads); Structural Stability Analysis.

* Dean, School of Engineering. Associate Fellow AIAA.

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Comment on "Hypersonic, Viscous Shock Layer with Chemical Nonequilibrium for Spherically Blunted Cones"

E. W. MINER* AND CLARK H. LEWIS†

Virginia Polytechnic Institute and State University, Blacksburg, Va.

IN a recent Note,[1] Kang and Dunn reported some of their numerical results for thin viscous shock-layer (TVSL) flows over spherically blunted cones. This Note was later supplemented

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* Assistant Professor, Department of Aerospace Engineering. Associate Member AIAA.

† Professor, Department of Aerospace Engineering, Associate Fellow AIAA.